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Invariant embedding treatment of phase randomisation and electrical noise at disordered surfaces

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Abstract. We derive the Fokker–Planck equation for the distribution of the phase of the reflection amplitude for a disordered conductor of length L, using the invariant embedding method. The limiting ($L = \infty$) stationary distribution in the strong-reflection regime is uniform provided that the localisation length is large compared with a Fermi wavelength. Next we study the joint distribution of the phase shift and of the time delay experienced by an incident electron in the conductor before being back-scattered. We obtain the explicit form of the marginal distribution for time delays for large L and strong reflections, using the uniformity of the phase distribution. We apply it to study the spectral density of the surface charge fluctuation noise. An approximate analytical calculation yields an $f^{-1/2(3-\alpha/2)}$ spectrum where $\alpha \geqslant 1$.

1. Introduction

This paper deals with two related aspects of first-principles theories of fluctuations of transport properties in mesoscopic systems due to Anderson localisation.

First we study the distribution of the phase θ of the amplitude reflection coefficient for a strongly reflecting one-dimensional conductor of length L. Strong reflection occurs when L is large compared with the localisation length L_c. Such a study is of interest for scaling theories of resistance and/or conductance fluctuations which usually regard the phase as a uniformly distributed independent random variable, as a result of successive scatterings by the random potential. Previous analyses of the phase distribution [1, 2] in specific models provided only partial support for the above assumption for weak disorder. However, an unambiguous justification of phase randomisation was found recently by the present author [3] for the weak-reflection or quasi-metallic domain ($L \ll L_c$), for $L \gg k_0^{-1}$. Here $\hbar k_0$ is the momentum of an incident electron. Moreover, both the uniformity of the phase distribution and the absence of significant correlations between the phase and the modulus of the reflection amplitude were demonstrated by a calculation of the exact low-order resistance moments [3], which differed from the random phase moments only by terms of order $(k_0L)^{-2} \ll 1$. The analysis in [3] is based on the method of invariant embedding which has been successfully applied in the context of resistance fluctuations [4–6]. The present study of the phase distribution in the insulating or strong-reflection regime ($L \gg L_c$) also uses the embedding method. More precisely, we shall obtain the explicit form of the stationary ($L = \infty$) solution of the

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Fokker–Planck equation associated with the invariant embedding equation for the reflection amplitude in the presence of the random potential.

An important consequence of phase randomness is the occurrence of surface current noise, as shown recently in [7]. In [7] it was observed that Anderson localised states in an infinite system become short-lived resonances near the surface of a bounded sample [7, 8] owing to their overlap with free particle states in the vacuum region. These Anderson resonances lead to resonant back-scattering with random time delays for different energy Fourier components of an incident wave-packet and, hence, to low-frequency current fluctuation noise. In [7] standard resonance scattering theory, where the phase is defined phenomenologically in terms of the position and the width of a resonance, was used and a $1/f$ noise spectrum was obtained. The second aspect considered in this paper is the discussion of the above electrical noise from the more fundamental invariant embedding point of view. The random phase θ and the time delay [7] $\tau = \hbar\, \mathrm{d}\theta/\mathrm{d}E$ are defined in terms of the characteristics of the trajectory followed by an incident electron in the disordered region before being reflected. These definitions allow us to study the joint distribution of τ and θ and, in particular, the marginal distribution of τ, which enters in the general expression in [7] for the low-frequency current spectral density in terms of phases. In an analytical approximation, our treatment yields a charge fluctuation noise of the form $f^{-(1/2)(3-\alpha/2)}$, where the constant parameter $\alpha \geqslant 1$ describes the growth rate with L of a typical value of τ. This form of the charge fluctuation noise reduces to the $1/f$ form [7] if $\alpha = 2$.

In § 2 we recall the coupled invariant embedding equations for the modulus $\sqrt{r}$ and the phase θ of the reflection amplitude for a one-dimensional disordered conductor of length L. By differentiation of the equation for θ, we obtain a further equation, depending on r and θ, for the delay time τ. With the assumption of a white-noise random potential the above stochastic equations are used in the strong-reflection limit ($r \sim 1$) to obtain the Fokker–Planck equations for the phase distribution $P_\theta(\theta, L)$ and for the joint distribution $P(\tau, \theta, L)$ of τ and θ. Explicit solutions for the stationary ($L = \infty$) phase distribution and for the marginal distribution of delay times, for large L, as well as the discussion of electrical noise are presented in § 3. Some remarks concerning correlation functions and connections with other work are given in § 4.

2. Stochastic equation for the reflection amplitude and associated Fokker–Planck equations

The invariant embedding equation for the complex reflection amplitude [5] $R(L) = \sqrt{r}\exp(\mathrm{i}\theta)$ for an electron incident at the edge $x = L$ of a one-dimensional conductor of length L may be split into the coupled equations

$$\mathrm{d}\rho/\mathrm{d}L = -[2V(L)/k_0]\sqrt{\rho(\rho+1)}\sin\theta \tag{1a}$$

$$\mathrm{d}\theta/\mathrm{d}L = 2k_0 - [2V(L)/k_0]\{1 + [(2\rho+1)/2\sqrt{\rho(\rho+1)}]\cos\theta\} \tag{1b}$$

for the phase θ and the dimensionless resistance ρ given by the Landauer [9] formula

$$\rho = r(1-r)^{-1}. \tag{2}$$

Here $E = k_0^2/2$ (in units such that $\hbar = m = 1$) is the energy of the incident electron and

$V(L)$ is the random potential which we take to be Gaussian, δ correlated and of mean zero:

$$\langle V(L)V(L')\rangle = V_0^2\delta(L-L') \qquad \langle V(L)\rangle = 0. \tag{3}$$

We are interested in the strong-reflection regime ($r \sim 1$, i.e. $\rho \to \infty$) where we shall also study the quantity

$$\tau = \mathrm{d}\theta/\mathrm{d}E \tag{4}$$

giving the time interval by which an incident electron is delayed in the disordered conductor before being reflected [7]. From (1*b*), we obtain

$$\mathrm{d}\tau/\mathrm{d}L = 2/k_0 + [2V(L)/k_0^3](1+\cos\theta + k_0^2\tau\sin\theta) \qquad r\sim 1. \tag{5}$$

The first term on the right-hand side of equation (1*b*) (and hence the corresponding term in equation (5)) has an obvious interpretation: $2k_0\,\mathrm{d}L$ is the phase change resulting from a return trip of an electron to a location a distance $\mathrm{d}L$ below the surface, in the absence of the random potential. The second term in (1*b*) then represents the change in the electron momentum (up to a factor of 2) due to the effect of the random potential. As mentioned in the previous section, the precise connection between this microscopic definition of phase changes and the definition of the phase in terms of the parameters of the resonant scattering model in [7] is not fully clear. The comparison in § 3 of the form of the phase distribution and of the form of the electrical noise for the two cases is therefore of particular interest.

We now derive Fokker–Planck equations successively for the joint distribution $P(\rho, \theta, L)$ of the coupled stochastic variables ρ, θ in equations (1*a*) and (1*b*) and for the joint distribution $P(\tau, \theta, L)$ of the variables τ, θ (for $\rho \to \infty$) obeying equations (1*b*) and (5). We follow a general procedure based on Van Kampen's [10] lemma.

In the general case of n stochastic variables $\{x_j\}$, $j = 1, 2, \ldots, n$, coupled by a white noise $w(\xi)$ evolving in a generalised 'time' (ξ) space, with

$$\langle w(\xi)w(\xi')\rangle = w_0^2\delta(\xi-\xi') \qquad \langle w(\xi)\rangle = 0 \tag{6}$$

the stochastic equations analogous to (1*a*) and (1*b*) are

$$\mathrm{d}x_i/\mathrm{d}\xi = F_i(\{x_i\}) + G_i(\{x_i\})w(\xi) \qquad i = 1, 2, \ldots, n. \tag{7}$$

The set $x \equiv \{x_i\}$ of x_i-values at a given 'time' represents a point in an n-dimensional phase space evolving according to equations (7). The form of the trajectory depends on the initial conditions and on the realisation of $w(\xi)$. Consider now an ensemble of systems in x-space corresponding to a given realisation of $w(\xi)$ but different initial conditions and evolving according to equations (7). This ensemble is described by a density $\rho(\{x_i\}, \xi)$, which obeys a continuity equation (stochastic Liouville equation)

$$\frac{\partial\rho(\{x_j\},\xi)}{\partial\xi} = -\sum_i \frac{\partial}{\partial x_i}[\![F_i(\{x_j\}) + w(\xi)G_i(\{x_j\})]\rho(\{x_j\},\xi)]\!]. \tag{8}$$

According to Van Kampen's [10] lemma, the joint probability density of the variables $\{x_i\}$ is given by the average of $\rho(\{x_j\}, \xi)$ over realisations of $w(\xi)$:

$$P(\{x_j\},\xi) = \langle\rho(\{x_j\},\xi)\rangle. \tag{9}$$

By averaging equation (8) giving the rate of change in ρ at a fixed point x, we obtain

$$\frac{\partial P(\{x_j\},\xi)}{\partial\xi} = -\sum_i \frac{\partial}{\partial x_i}[F_i(\{x_j\})P(\{x_j\},\xi) + G_i(\{x_j\})\langle w(\xi)\rho(\{x_j\},\xi)\rangle] \tag{10}$$

where the form of the left-hand side follows from the comparison of the average of (8) with the ξ-derivative of the average of its first integral. From Novikov's [11] formula we then have

$$\langle w(\xi)\rho(\{x_j\}, \xi)\rangle = (w_0^2/2)\langle \delta\rho(\{x_j\}, \xi)/\delta w(\xi)\rangle \tag{11}$$

where the 'equal-time' correlation function $\delta\rho(\{x_j\}, \xi)/\delta w(\xi)$ is obtained explicitly by differentiation of the first integral in equation (8), recalling that $\delta\rho(\{x_j\}, \xi)/\delta w(\xi') = 0$ for $\xi' > \xi$ because of causality. In this way we finally obtain

$$\frac{\partial P(\{x_j\}, \xi)}{\partial \xi} = -\sum_i \frac{\partial}{\partial x_i}[F_i(\{x_j\})P(\{x_j\}, \xi)] + \frac{w_0^2}{2}\sum_{i,j}\frac{\partial}{\partial x_i}\left[\!\left[G_i(\{x_k\})\frac{\partial}{\partial x_j}[G_j(\{x_k\})P(\{x_k\}, \xi)]\right]\!\right] \tag{12}$$

which may be readily reduced to the more standard form [12] of the Fokker–Planck equation for white-noise processes with several stochastic variables, by rewriting the diffusion term in (12) as

$$\frac{w_0^2}{2}\sum_{i,j}\left[\frac{\partial^2}{\partial x_i\,\partial x_j}(G_iG_jP) - \frac{\partial}{\partial x_i}\left(\frac{\partial G_i}{\partial x_j}(G_jP)\right)\right]$$

and grouping the last term with the drift term in equation (12).

Now, in the special case of the two stochastic variables ρ and θ defined by equations (1*a*) and (1*b*), equation (12) yields

$$\begin{aligned}\partial P/\partial l = {} & -2k_0L_c\,\partial P/\partial\theta + 2\sin^2\theta\,(\partial/\partial\rho)\big(\rho(\rho+1)\,\partial P/\partial\rho + (\rho+\tfrac{1}{2})P \\ & + [\sqrt{\rho(\rho+1)}/\sin\theta](\partial/\partial\theta)[\![\{1 + [(2\rho+1)/2\sqrt{\rho(\rho+1)}]\cos\theta\}P]\!]\big) \\ & + 2(\partial/\partial\theta)[\{1 + [(2\rho+1)/2\sqrt{\rho(\rho+1)}] \\ & \times\cos\theta\}\big(\sin\theta\,(\partial/\partial\rho)[\sqrt{\rho(\rho+1)}P] \\ & + (\partial/\partial\theta)[\![\{1 + [(2\rho+1)/2\sqrt{\rho(\rho+1)}]\cos\theta\}P]\!]\big)]\end{aligned} \tag{13}$$

where $l = L/L_c$ and $L_c = k_0^2/V_0^2$ is the localisation length. In the special case where ρ and θ are arbitrarily assumed to be independent and the distribution of θ is taken to be uniform between 0 and 2π, we have $P(\rho, \theta) = (2\pi)^{-1}P_\rho(\rho, L)$. In this case averaging over θ in equation (13) to introduce the marginal resistance distribution $P_\rho(\rho, L) = \int_0^{2\pi} \mathrm{d}\theta\, P(\rho, \theta)$ readily yields the familiar equation

$$\partial P_\rho/\partial l = (\partial/\partial\rho)[\rho(\rho+1)\,\partial P_\rho/\partial\rho] \tag{14}$$

which has been studied in previous work [4, 6]. Our interest here lies in the marginal phase distribution $P_\theta(\theta, L) = \int_0^\infty \mathrm{d}\rho\, P(\rho, \theta, L)$ in the strong-reflection limit ($\rho \to \infty$) which, to our knowledge, has not been discussed before. Integration over ρ of the limiting form for $\rho \gg 1$ of equation (13), with the boundary conditions $P(\rho\to\infty, \theta, L) = 0$ and $\partial P/\partial\rho|_{\rho\to\infty} = 0$, yields

$$\begin{aligned}\partial P_\theta(\theta, L)/\partial l = {} & -2k_0L_c\,\partial P_\theta(\theta, L)/\partial\theta \\ & + 2(\partial/\partial\theta)\{(1+\cos\theta)(\partial/\partial\theta)[(1+\cos\theta)P_\theta(\theta, L)]\}\end{aligned} \tag{15}$$

whose solutions will be discussed in § 3.

Finally we use equation (12) to obtain the Fokker–Planck equation for the joint distribution $P \equiv P(\tau, \theta, L)$ (for $r \sim 1$) of the variables θ and τ defined by equations (1*b*) and (5). Using the large-ρ limit of (1*b*), we get

$$\begin{aligned}\partial P/\partial l = & -2k_0L_c\,\partial P/\partial\theta - (2L_c/k_0)(\partial P/\partial\tau) + 2(\partial/\partial\theta)\\ & \times\{(1+\cos\theta)(\partial/\partial\theta)[(1+\cos\theta)P]\} - (2/k_0^2)(\partial/\partial\theta)\\ & \times\{(1+\cos\theta)(\partial/\partial\tau)[(1+\cos\theta+k_0^2\tau\sin\theta)P]\} - (2/k_0^2)(\partial/\partial\tau)\\ & \times\{(1+\cos\theta+k_0^2\tau\sin\theta)(\partial/\partial\theta)[(1+\cos\theta)P]\} + (2/k_0^4)(\partial/\partial\tau)\\ & \times\{(1+\cos\theta+k_0^2\tau\sin\theta)(\partial/\partial\tau)[(1+\cos\theta+k_0^2\tau\sin\theta)P]\}.\end{aligned} \tag{16}$$

Here again we are not interested in the full information contained in the joint distribution $P(\tau, \theta, L)$, but only in the marginal distribution of time delays τ, i.e.

$$P_\tau(\tau, L) = \int_0^{2\pi} \mathrm{d}\theta\, P(\tau, \theta, L). \tag{17}$$

Since τ is a derivative (equation (4)), we expect it to fluctuate much more rapidly than the random variable θ itself. Therefore we shall treat τ and θ as independent and, as in the above discussion of the resistance distribution [4, 6], we assume θ to be uniformly distributed, i.e. $P(\tau, \theta, L) = (2\pi)^{-1}P_\tau(\tau, L)$. Then, after averaging both sides of equation (16) over θ to introduce the distribution (17), we obtain

$$\partial P_\tau/\partial l = 2P_\tau + (4\tau - 2L_c/k_0)\partial P_\tau/\partial\tau + (3/k_0^4 + \tau^2)\partial^2 P_\tau/\partial\tau^2. \tag{18}$$

A similar equation for the distribution of time delays has been derived previously in [13]. The assumption of a uniform phase distribution is supported by our detailed study of the latter distribution in § 3. The explicit solution of equation (18) will be used to study electrical noise, as discussed above.

3. Detailed results

3.1. Stationary phase distribution

The solution of equation (15) may be found analytically in the stationary regime, $L \to \infty$, where $\partial P_\theta/\partial l = 0$. In this case, equation (15) is solved by

$$P_\theta(\theta, l\to\infty) = \frac{C}{2(1+\cos\theta)}\exp[k_0L_cI(\theta)]\int_0^\theta \mathrm{d}\theta'\frac{\exp[-k_0L_cI(\theta')]}{1+\cos\theta'} \tag{19}$$

where

$$I(\theta) = \int^\theta \mathrm{d}\theta'\frac{1}{(1+\cos\theta')^2} = \frac{1}{3}\tan\left(\frac{\theta}{2}\right)\left(2+\frac{1}{\cos^2(\theta/2)}\right) \tag{20}$$

and C is the arbitrary constant appearing in the first integral of the right-hand side of (15) which is determined by normalisation: $\int_0^{2\pi}\mathrm{d}\theta P_\theta(\theta, l\to\infty) = 1$. For $k_0L_c \gg 1$, i.e. for L_c well above the Ioffe–Regel limit (k_0^{-1} equals approximately the Fermi wavelength of the leads) we may give (19) in terms of an expansion in powers of $1/k_0L_c$ obtained by successive partial integrations:

$$P_\theta(\theta, l\to\infty) = (1/2\pi)[1+1/2(k_0L_c)^2]^{-1}[1-(1/k_0L_c)\sin\theta(1+\cos\theta) - [1/(k_0L_c)^2](1+\cos\theta)^2(1+\cos\theta-3\sin^2\theta)+\cdots] \tag{21}$$

where normalisation has yielded $C = -\pi^{-1}k_0L_c[1+2^{-1}(k_0L_c)^{-2}]^{-1}$. We demonstrate therefore that in the strong-reflection limit the stationary phase distribution tends indeed to a uniform distribution for $k_0L_c \gg 1$. A similar conclusion concerning the form of the phase distribution was reached previously [3] for the opposite limit, namely the low-reflection or quasi-metallic domain $L \ll L_c$, for $k_0L \gg 1$. We note that in a different context the distribution of the phase of electromagnetic waves back-scattered by a random refractive medium has been studied [1] and found to be relatively uniform except near the edges of the 2π interval. On the other hand, the results in [8] indicate that for the localised resonances model the phase distribution is generally non-uniform, except near odd multiples of $\pm\pi/2$.

3.2. *Electrical surface noise*

For an incident electron of energy E the average surface current spectral density $S_J(\omega)$ at frequency ω reduces approximately to the form [7, 13]

$$S_J(\omega) \simeq S_\infty(E)\langle\sin^2\{[\theta(E+\omega)-\theta(E)]/2\}\rangle \tag{22}$$

in the strong-reflection limit ($r \sim 1$). Here, the averaging is over the disorder and the frequency dependence of $S_\infty(E)$ is much weaker than that of the remaining factor. At low frequencies, we have

$$\theta(E+\omega)-\theta(E) \simeq \omega\,\mathrm{d}\theta/\mathrm{d}E = \omega\tau \tag{23}$$

provided that the otherwise arbitrary quantity $\omega\tau$ represents the leading term, which requires that $\omega \ll 2|\mathrm{d}(\ln\tau)/\mathrm{d}E|^{-1}$. By inserting (23) into (22), we have

$$S_J(\omega) \simeq S_\infty(E)\langle\sin^2(\omega\tau/2)\rangle \simeq S_\infty(E)\int_{\tau_{\min}}^{\infty}\mathrm{d}\tau\, P_\tau(\tau, L)\sin^2\left(\frac{\omega\tau}{2}\right) \tag{24}$$

where the lower limit $\tau_{\min}$ for the integral only means that we take ω to be sufficiently large that the uninteresting contribution (proportional to ω^2) from the domain $0<\tau<\tau_{\min}\sim 2/\omega$ is negligible.

For the further analysis of equation (24) we require the delay time distribution $P_\tau(\tau, L)$, determined by equation (18). To obtain the explicit form of $P_\tau(\tau, L)$ for large but finite L, we first derive the form of the moments

$$\tau_n \equiv \langle\tau^n\rangle = \int_0^\infty \mathrm{d}\tau\,\tau^n P_\tau(\tau, L) \qquad n=0,1,2,\ldots. \tag{25}$$

Recursion relations for the moments are obtained by multiplying both sides of (18) by τ^n and integrating over τ. After some partial integrations, using the boundary condition $P_\tau(\tau\to\infty, L)=0$, we get

$$\partial\tau_n/\partial l = n(n-1)\tau_n + 2n(L_c/k_0)\tau_{n-1} + (3/k_0^4)n(n-1)\tau_{n-2} \qquad n=0,1,2,\ldots \tag{26}$$

where $\tau_0 = 1$ by normalisation. In the strong-reflection regime, $L \gg L_c$, the solutions for the moments are rapidly growing functions of the order $n(\tau_n \gg \tau_{n-1}$, etc). Thus, for $n > 1$, we obtain

$$\tau_n \sim \exp[n(n-1)l] \tag{27}$$

while, for $n = 1$, we have

$$\langle\tau\rangle = 2L/k_0. \tag{28}$$

Equation (28) reveals a surprising property, namely that in the strong-reflection regime the mean value of the delay time coincides with the time required for an incident electron to execute a return trip across the sample, in the absence of randomness. On the other hand, it follows from (27) and (28) that the delay time is not a self-averaging quantity since for example the RMS deviation grows more rapidly than the mean.

The moments (27) enable us to obtain the asymptotic form of $P_\tau(\tau, L)$ from the characteristic function

$$\varphi(k) = \sum_n \frac{(\mathrm{i}k)^n}{n!}\tau_n. \tag{29}$$

Here the summation over the moments may be performed in closed form using the identity

$$\exp[n(n-1)l] = (4\pi l)^{-1/2}\exp[-(2n+\tfrac{1}{4})l]\int_{-\infty}^{\infty}\mathrm{d}x\exp[(n+\tfrac{1}{2})x]\exp\left(-\frac{x^2}{4l}\right) \tag{30}$$

and we shall ignore higher-order effects due to the difference between (27) and the correct form (28) for τ_1. In this way we get

$$P_\tau(\tau, L) = (2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{d}k\exp(-\mathrm{i}\,k\tau)\varphi(k)$$

$$\simeq (4\pi l)^{-1/2}\tau^{-1}\exp\left(-\frac{1}{4l}(\ln\tau + l)^2\right) \tag{31}$$

which has the form of the weak tail of a normalised Gaussian for the variable $\ln\tau \gg 1$. We note that the function (31) differs from the corresponding asymptotic distribution of resistance in an essential way; the maximum of the Gaussian distribution of the logarithm of resistance [4] occurs at the value $l \gg 1$ (instead of $-l$), i.e. within the asymptotic domain.

We now evaluate the current spectral density by inserting (31) into (24) which we then rewrite in the form

$$S_J(\omega) = S_\infty(E)\frac{\exp(-l/4)}{\sqrt{4\pi l}}\int_{\tau_{\min}}^{\infty}\mathrm{d}\tau\frac{\sin^2(\omega\tau/2)}{\tau^{(1/4l)\ln\tau+3/2}}. \tag{32}$$

In order to approximate the frequency dependence of $S_J(\omega)$ in closed form, we replace the slowly varying logarithmic exponent in the integrand by a typical value. From the form (27) of the nth-order ($n > 1$) moment it is clear that typical values of $\ln\tau$ are of the form $\ln\tau \sim \alpha l$, where α is constant, $\alpha \gg 1$. With this approximation it follows that

$$S_J(\omega) \sim \omega^{(1/2)(1+\alpha/2)} \tag{33}$$

which implies a $1/\omega^\beta$ noise, $\beta = \frac{1}{2}(3 - \alpha/2)$, for surface charge (voltage) fluctuations.

We thus find that the low-frequency surface charge fluctuation noise varies with an exponent $\beta \leqslant \frac{5}{4}$, which is generally different from the $1/f$ form obtained in [7] for the localised resonances model [14].

4. Concluding remarks

In this paper we have discussed the distribution of the phase of the reflection coefficient and the distribution of the time delay for reflection, together with the resulting electrical noise, in the strong-reflection regime. Our explicit study of the phase distribution in the stationary limit strongly supports previous analyses of scaling properties of the resistance and of the transmission coefficient [4, 6, 15–18] based on uniformly distributed phases. Our first-principles study of electrical noise due to randomly delayed reflection of electrons at the surface leads to an approximate $1/\omega^{\beta}$ voltage noise ($\beta \leqslant \frac{5}{4}$). This has a more complicated form than the $1/f$ noise predicted recently in [7] for a model of Anderson localised resonances.

An alternative treatment of the above electrical noise would consist in finding directly the phase difference $\varphi = \theta(E+\omega) - \theta(E)$ distribution, at neighbouring energies in equation (22). This distribution may be studied starting from the stochastic equation for φ obtained from (1*b*) by defining the variables φ and $\theta(E)$, instead of $\theta(E+\omega)$ and $\theta(E)$. A Fokker–Planck equation for the marginal distribution $P_{\varphi}(\varphi, L)$ may then be obtained by averaging uniformly over absolute phases $\theta(E)$, as usual. Unfortunately, the resulting equation, which reduces to (18) for $\varphi \ll 1$, is generally intractable analytically. Therefore the approach in § 3 which, however, does not assume $\varphi \ll 1$ appears to be more judicious here. Note also that in that case the distribution of φ given by (23) is simply

$$P_{\varphi}(\varphi, L) = (1/\omega) P_{\tau}(\tau = \varphi/\omega, L) \tag{34}$$

which is valid for sufficiently small ω.

The distributions of phase differences $\varphi = \theta(E') - \theta(E)$ at slightly different energies E' and E, based on averaging over the remaining phase variable $\theta(E)$, have been studied in [19–22] for the purpose of analysing correlation functions. In fact, our results for the moments of the delay time distribution may be readily used to study the phase correlation function

$$\langle \theta(E')\theta(E)\rangle - \langle \theta(E')\rangle\langle \theta(E)\rangle \tag{35}$$

and, in particular, the phase decorrelation length [21, 22], which determines the dynamical conductivity. Following the results in § 3.1, we assume $\theta(E)$ to be uniformly distributed between 0 and 2π and we have

$$\begin{aligned}\langle[\theta(E') - \theta(E)]^2\rangle &= -2\langle\theta(E')\theta(E)\rangle + 8\pi^2/3 \\ &\simeq (\Delta E)^2\langle\tau^2(E)\rangle \qquad \Delta E = E' - E.\end{aligned}$$

The correlation function (35) then takes the form

$$\langle\theta(E')\theta(E)\rangle - \langle\theta(E')\rangle\langle\theta(E)\rangle = \pi^2/3 - [(\Delta E)^2/2]\langle\tau^2(E)\rangle \tag{36}$$

where we insert the exact solution of (26) for $\tau_2 \equiv \langle\tau^2\rangle$ that vanishes for $l = 0$, namely

$$\begin{aligned}\langle\tau^2\rangle &= (2/k_0^4)[(\tfrac{3}{2} + k_0^2 L_c^2)\exp(2l) - \tfrac{3}{2} - 2k_0^2 L_c^2(l + \tfrac{1}{2})] \\ &\simeq (2/k_0^4)(\tfrac{3}{2} + k_0^2 L_c^2)\exp(2l) \qquad l \gg 1.\end{aligned} \tag{37}$$

The phase decorrelation length l^* is the particular length at which (36) vanishes and from (36) and (37) we get

$$L^* = L_c \ln[\sqrt{2}\pi k_0^2/\sqrt{3(3 + 2k_0^2 L_c^2)}|\Delta E|]. \tag{38}$$

This expression for the decorrelation length is similar to an expression obtained in [22] (see their equation (3.10)). As discussed in [22], L^* is related to the microscopic phase correlation length introduced in [21] and used to determine the dynamical conductivity $\sigma(\omega)$ and, in particular, to justify the Mott–Davis [23] expression for the hopping conductivity:

$$\sigma(\omega) \propto (\omega\tau^*)^2 \ln^2(\omega\tau^*) \tag{39}$$

where τ^* is the elastic scattering time.

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[14] The analysis of surface charge fluctuation noise in [13] was based on a stationary ($L = \infty$) solution obtained by setting $\partial P_\tau/\partial l = 0$ in the form of the Fokker–Planck equation (18) used in [13] (their equation (6)). A particular normalised solution was chosen such that the first integral of the right-hand side of their equation (6) vanishes (the so-called potential case). We find, however, that this type of stationary solution, which in the case of (18) takes the form (for $k_0 L_c \gg 1$)

$$P_\tau(\tau, \infty) \simeq (2k_0^3 L_c/3)[\![\{\exp[(2k_0 L_c/\sqrt{3})(\tan^{-1} u - \pi/2)]\}/(1 + \hat{u}^2)]\!] \qquad u = \tau k_0^2/\sqrt{3}$$
$$\simeq (2L_c/k_0)(1/\tau^2) \qquad \tau \to \infty$$

is somewhat different from the stationary limit, at $\tau = \exp l$,

$$P_\tau(\tau, \infty) = (1/\sqrt{4\pi})(1/\tau^2\sqrt{\ln \tau})$$

of our general normalised asymptotic solution (31) for finite L. In particular, the distribution in [13] yields a logarithmically divergent mean delay time instead of the correct expression (28). In any case, we believe that the asymptotic distribution for finite L, equation (31), leads to a much more accurate prediction for the noise spectrum than does the stationary limit above.

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